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Varieties of Abelian mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$

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ABSTRACT: We study 3d mirror symmetry with loop operators, Wilson loop and Vortex loop, and multi-flavor mirror symmetry through utilizing the $\mathbb{RP}^2 \times \mathbb{S}^1$ index formula. The key identity which makes the above description work well is the mod 2 version of the Fourier analysis, and we study such structure, the S-operation in the context of a $SL(2, \mathbb{Z})$ action on 3d SCFTs. We observed that two types of the parity conditions basically associated with gauge symmetries which we call \mathcal{P} -type and \mathcal{CP} -type are interchanged under mirror symmetry. We will also comment on the T-operation.

KEYWORDS: Supersymmetry and Duality, Supersymmetric gauge theory, Extended Supersymmetry

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1 Introduction

Three-dimensional $\mathcal{N} = 2$ mirror symmetry [1–4] is one of the nontrivial classes of dualities among quantum field theories, which states that two distinct theories defined in the UV regime flow into an identical IR fixed point. In order to test such dualities, exact physical quantities protected by supersymmetry are often useful and provide to us understandings of the duality in mathematically well-defined languages.

In this paper, we deepen the study of 3d Abelian mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$ performed in [5, 6]. We show that the correspondences of loop operators and the multi-flavor version of mirror symmetry which were studied on other three-dimensional manifolds [7–9] also work even on $\mathbb{RP}^2 \times \mathbb{S}^1$ in terms of the index [10]. The key mathematical identity to demonstrate them is a *mod 2* delta function’s formula:

$$\frac{1}{2} \sum_{a=0,1} e^{i\pi a(A-B)} = \delta_{A,B}^{\text{mod } 2}. \quad (1.1)$$

This is nonvanishing only if $A - B \equiv 0 \pmod{2}$. See (2.23) for details. The mod 2 structure can be regarded as an algebraic representation of the \mathbb{RP}^2 unorientable structure.

As considered somewhat systematically in [6], in order to define supersymmetric gauge theories on $\mathbb{RP}^2 \times \mathbb{S}^1$, there are *two* types of parity conditions for the vector multiplet consistent with the unorientable structure which we call \mathcal{P} -type and \mathcal{CP} -type. The covariant derivative under each parity condition transforms as

$$\bullet \text{ } \mathcal{P}\text{-type vector multiplet } \ni A_\mu: \quad (\partial_\mu - iQ A_\mu) \rightarrow \pm(\partial_\mu - iQ A_\mu). \quad (1.2)$$

$$\bullet \text{ } \mathcal{CP}\text{-type vector multiplet } \ni A_\mu: \quad (\partial_\mu - iQ A_\mu) \rightarrow \pm(\partial_\mu + iQ A_\mu). \quad (1.3)$$

The former simply brings the parity (\mathcal{P}) effect, while the latter accompanies charge conjugation (\mathcal{C}), $+Q \leftrightarrow -Q$, with parity.

In addition, we observe that it is possible to calculate *e*-charged *BPS-Wilson loop* W_e and *m*-vorticity *BPS-Vortex loop* V_m . As well known, the Wilson loop is an order parameter of the system. On the other hand, the Vortex loop, which is a codimension-2 defect in 3d theories, is a disorder parameter of the system and is defined by not a standard operator insertion but singular behavior of the fields around the loop [8, 9, 11]. In fact, there is one simple way [8] to describe the Vortex loop by making use of *S-operation* of a $\text{SL}(2, \mathbb{Z})$ group in 3d CFTs [12]. We systematically find the correspondences of these loop operators based on that fact in the context of 3d $\mathcal{N} = 2$ mirror symmetry with a single matter $N_f = 1$:

$$\mathcal{P}\text{-type Gauge } \left\{ \begin{array}{c} \text{Wilson} \\ \text{Vortex} \end{array} \right\} \text{ loop} \quad \Leftrightarrow \quad \mathcal{CP}\text{-type Global } \left\{ \begin{array}{c} \text{Vortex} \\ \text{Wilson} \end{array} \right\} \text{ loop} \quad (1.4)$$

$$\mathcal{CP}\text{-type Gauge } \left\{ \begin{array}{c} \text{Wilson} \\ \text{Vortex} \end{array} \right\} \text{ loop} \quad \Leftrightarrow \quad \mathcal{P}\text{-type Global } \left\{ \begin{array}{c} \text{Vortex} \\ \text{Wilson} \end{array} \right\} \text{ loop} \quad (1.5)$$

The Wilson loop on one side is mapped to the Vortex loop on the dual side as seen in the general discussion [8, 9] and also in [11] for $\mathcal{N} = 4$. The remarkable point in this duality is to interchange \mathcal{P} -type and \mathcal{CP} -type for the dynamical (Gauge) or background (Global) gauge field. This can be regarded as the non-trivial effect originated from the unorientable structure.

We also find that the mirror symmetry exchanges \mathcal{P} and \mathcal{CP} in many ways. One example is multi-flavor mirror symmetry, the duality between the SQED with N_f flavors (SQED_{N_f}) and $\text{U}(1)^{N_f-1}$ quiver gauge theory, which is realized by the following two ways:

$$\begin{aligned} \mathcal{P}\text{-SQED}_{N_f} &\Leftrightarrow \mathcal{CP}\text{-U}(1)^{N_f-1} \text{ gauge theory,} \\ \mathcal{CP}\text{-SQED}_{N_f} &\Leftrightarrow \mathcal{P}\text{-U}(1)^{N_f-1} \text{ gauge theory,} \end{aligned} \quad (1.6)$$

where leading letters represent the parity type for the vector multiplet in the gauge group. These are new properties in 3d theories equipped with geometrical \mathbb{Z}_2 -structure.

The rest of this paper is organized as follows. In section 2, we briefly review the exact results by localization techniques applied to path integrals of Abelian SUSY gauge theories on $\mathbb{RP}^2 \times \mathbb{S}^1$ and provide recipe for the computation and the usage of S-operation. In section 3, we show that the correspondences of the BPS loop operators (1.4) and (1.5) in single-flavor mirror symmetry by utilizing S-operation on $\mathbb{RP}^2 \times \mathbb{S}^1$. In section 4, we check multi-flavor mirror symmetry (1.6) by using the localization recipe and the gauging procedure of the global symmetry. We comment on open questions in section 5. In appendix, we summarize the bases of the BPS loop operators which are \mathbb{Z}_2 -invariant on \mathbb{RP}^2 .

2 Features of SUSY gauge theory on $\mathbb{RP}^2 \times \mathbb{S}^1$

2.1 Lightning review on localization computation on $\mathbb{RP}^2 \times \mathbb{S}^1$

This subsection is dedicated to the review of [5, 6]. What we consider is \mathbb{RP}^2 index,

$$\mathcal{I}^{\mathbb{RP}^2}(q) = \text{Tr}_{\mathcal{H}^{\mathbb{RP}^2}} \left[(-1)^{\hat{F}} q^{\frac{1}{2}(\hat{H} + \hat{j}_3)} \right], \quad (2.1)$$

where q is fugacity and \hat{F} , \hat{H} , \hat{j}_3 are fermion number, Hamiltonian, 3rd angular momentum. $\mathcal{H}^{\mathbb{RP}^2}$ is the theory's \mathbb{Z}_2 -divided Hilbert space. In order to compute this quantity, we rewrite this definition by using path integral over the supersymmetric multiplets,

$$\text{Vector multiplet } V = (\sigma, \lambda, \bar{\lambda}, A_\mu), \quad \text{Matter multiplet } \Phi = (\phi, \psi), (\bar{\phi}, \bar{\psi}), \quad (2.2)$$

where ϕ and $\bar{\phi}$ are complex scalars, σ is real scalar, ψ and λ are spinors, A_μ is gauge field.¹ The effect of considering \mathbb{Z}_2 -divided Hilbert space should emerge in the boundary conditions for the fields under the reflection, and the (2.1) becomes

$$\mathcal{I}^{\mathbb{RP}^2}(q) = \int_{\mathbb{RP}^2 \times \mathbb{S}^1} \mathcal{D}\Phi \mathcal{D}V e^{-S[\Phi, V]}, \quad (2.3)$$

with appropriate boundary conditions as explained below. In order to represent arbitrary theory's $\mathcal{I}^{\mathbb{RP}^2}(q)$, we use quiver diagram.

Vector multiplet. Compared with the conventional notation, $\mathbb{RP}^2 \times \mathbb{S}^1$ quiver diagram is slightly different. One important difference is that we have *two types* of the vector multiplets $V^{(\mathcal{P})}$ and $V^{(\mathcal{CP})}$ [6], and represent them as nodes with characters \mathcal{P} or \mathcal{CP} :

$$\begin{aligned} V^{(\mathcal{P})} : \text{dynamical} &= \bigcirc_{\mathcal{P}}, \quad \text{background} = \bigcirc_{s^\pm, \theta} \text{ where } \begin{pmatrix} s^+ = 0, \\ s^- = 1, \end{pmatrix} \theta \in [0, 2\pi] \Big), \\ V^{(\mathcal{CP})} : \text{dynamical} &= \bigcirc_{\mathcal{CP}}, \quad \text{background} = \bigcirc_{s, \theta_\pm} \text{ where } \begin{pmatrix} s \in \mathbb{Z}, & \theta_+ = 0 \\ & \theta_- = \pi \end{pmatrix}. \end{aligned} \quad (2.4)$$

s_\pm represents holonomy along nontrivial path in $\pi_1(\mathbb{RP}^2)$, and s corresponds to Dirac monopole charge² on \mathbb{RP}^2 . θ and θ_\pm are holonomy along \mathbb{S}^1 .

Matter multiplet. We represent matter multiplet degrees of freedom by a rectangular node. As same as the case of vector multiplet, we have basically two possible matter multiplets, Φ_s and Φ_d , a singlet and a doublet:

$$\begin{aligned} \Phi_s : \text{dynamical} &= \boxed{\Phi}, \\ \Phi_d : \text{dynamical} &= \boxed{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}. \end{aligned} \quad (2.5)$$

For more detail, see [6].

¹Vector multiplet V include gauge field A_μ , so the theory with V becomes gauge theory automatically. Throughout this paper, we only consider $U(1)$ gauge symmetry.

² $s_\pm = B_\pm$ and $s = B/2$ in [6] notation.

Gauge coupling. We represent gauge coupling between matter and vector multiplets with arrow \rightarrow connecting the two nodes. The number of arrowheads means charge \mathbf{Q} , and the direction of the arrow means the sign(\mathbf{Q}). To apply (2.1), we should make theory \mathbb{Z}_2 -invariant at least classically, and the possible couplings are as follows:

$$\boxed{\Phi} \xrightarrow{\mathbf{Q}} (\mathcal{P}), \quad \boxed{\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}} \xrightarrow{\mathbf{Q}} (\mathcal{P}), \quad \boxed{\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}} \xrightarrow{\mathbf{Q}} (\mathcal{CP}). \quad (2.6)$$

Note that matter singlet cannot couple to \mathcal{CP} -type because it includes charge conjugation \mathcal{C} , and singlet is not invariant under the \mathcal{C} . Of course, one can couple the background vector multiplets in (2.4) in these way, and in such case, we will use dotted arrow $\cdots \rightarrow$.

Topological coupling. We can define \mathbb{Z}_2 -invariant topological coupling only between \mathcal{P} -type vector multiplet and \mathcal{CP} -type vector multiplet,

$$(\mathcal{P}) \cdots \text{BF} \cdots (\mathcal{CP}), \quad (2.7)$$

where we use the index “BF” as positive coupling. If the coupling is negative, we use the index “−BF”. This coupling will be important in later discussion. One can consider one of them as background factor multiplet, too.

Superpotential. We can even turn on a supersymmetric potential among matter multiplets called superpotential if the interaction does not break the \mathbb{Z}_2 -invariance. However, the final result does not depend on the form of superpotential, so we neglect them.

For example, if we have the following quiver diagram,

$$\boxed{\Phi} \xrightarrow{\mathbf{Q}} (\mathcal{P}) \cdots \text{BF} \cdots (s, \theta_{\pm}) \quad (2.8)$$

then, what we calculate is the following quantity:

$$\mathcal{I}^{\mathbb{RP}^2}(q; s, \theta_{\pm}) = \int_{\mathbb{RP}^2 \times \mathbb{S}^1} \mathcal{D}\Phi \mathcal{D}V^{(\mathcal{P})} e^{-S}, \quad (2.9)$$

where $S = \int_0^\beta dy \int d^2x \sqrt{g} (\mathcal{L}_{\text{mat}} + \mathcal{L}_{\text{U}(1)} + \mathcal{L}_{\text{top}})$ and

$$\begin{aligned} \mathcal{L}_{\text{mat}} &= -i(\bar{\psi}\gamma^\mu[\nabla_\mu - i\mathbf{Q}A_\mu^{(\mathcal{P})}]\psi) + i\mathbf{Q}(\bar{\psi}\sigma^{(\mathcal{P})}\psi) - i\mathbf{Q}\bar{\phi}(\bar{\lambda}^{(\mathcal{P})}\psi) - \frac{i(2\Delta-1)}{2}(\bar{\psi}\gamma_y\psi) + \bar{F}F + i\mathbf{Q}(\bar{\psi}\lambda^{(\mathcal{P})})\phi \\ &\quad + |(\partial_\mu - i\mathbf{Q}A_\mu^{(\mathcal{P})})\bar{\phi}|^2 + \bar{\phi}(\mathbf{Q}\sigma^{(\mathcal{P})})^2\phi + i\mathbf{Q}\bar{\phi}D^{(\mathcal{P})}\phi - (2\Delta-1)\bar{\phi}[\partial_y - i\mathbf{Q}A_y^{(\mathcal{P})}]\phi - \Delta(\Delta-1)\bar{\phi}\phi, \\ \mathcal{L}_{\text{U}(1)} &= +\frac{1}{2}(F_{\mu\nu}^{(\mathcal{P})})^2 + (D^{(\mathcal{P})})^2 + (\partial_\mu\sigma^{(\mathcal{P})})^2 + \epsilon^{\mu\nu\rho}\sigma^{(\mathcal{P})}F_{\mu\nu}^{(\mathcal{P})} + (\sigma^{(\mathcal{P})})^2 + i\bar{\lambda}^{(\mathcal{P})}\gamma^\mu\nabla_\mu\lambda^{(\mathcal{P})} - \frac{i}{2}\bar{\lambda}^{(\mathcal{P})}\gamma_y\lambda^{(\mathcal{P})}, \\ \mathcal{L}_{\text{top}} &= \epsilon^{\mu\nu\rho}F_{\mu\nu}^{(\mathcal{P})}a_\rho(s, \theta_{\pm}) - 2sD^{(\mathcal{P})}, \end{aligned}$$

We write here the *auxiliary fields* $F, \bar{F}, D^{(\mathcal{P})}$, and after integrating out them, we get a nontrivial potential for ϕ . The Δ is an arbitrary parameter called *R-charge* for the matter multiplet. In the final line \mathcal{L}_{top} , we use the following configuration parametrized by (s, θ_{\pm})

$$a(s, \theta_{\pm}) = s(\cos\theta - 1)d\varphi + \frac{\theta_{\pm}}{2\pi}dy, \quad (\vartheta, \varphi): \mathbb{RP}^2 \text{ coordinates}. \quad (2.10)$$

Recipe for the index from localization computation. Thanks to supersymmetric localization argument, it turns out that the path integral can be calculated *exactly* by considering up to one loop. More precisely speaking, the Gaussian calculation is performed around configurations giving $S = 0$, and all of these contributions should be summed (or integrated) over. The result is summarized in section 2.2 of [6] as simple rule. By using the rule, we can calculate the path integral weighted by the theory characterized by a quiver diagram. The rule is composed by following steps:

- If there is node with rigid line, just replace it as sum of possible dotted objects

$$\left(\dots \longrightarrow \textcircled{\mathcal{P}} \right) = q^{+\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{s^\pm=0,1} \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\dots \longrightarrow \textcircled{s^\pm, \theta} \right), \quad (2.11)$$

$$\left(\dots \longleftrightarrow \textcircled{c\mathcal{P}} \right) = q^{-\frac{1}{8}} \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{s \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_\pm=0, \pi} \left(\dots \longleftrightarrow \textcircled{s, \theta_\pm} \right), \quad (2.12)$$

- Then, the resultant quiver diagram is represented by only dotted lines and dotted circler nodes. We regard it as the integrand or summand in (2.11) or (2.12) by using the following rules

$$\begin{aligned} & \textcircled{s_1^\pm, \theta_1} \xrightarrow[Q_1]{\textcircled{\Phi}} \textcircled{s_2^\pm, \theta_2} \\ &= \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{i(Q_1\theta_1+Q_2\theta_2)}{4}} \right)^{+1} \frac{(e^{-i(Q_1\theta_1+Q_2\theta_2)} q^{\frac{(2-\Delta)}{2}}; q^2)_\infty}{(e^{+i(Q_1\theta_1+Q_2\theta_2)} q^{\frac{(0+\Delta)}{2}}; q^2)_\infty} & \text{for } (-1)^{Q_1 s_1^\pm + Q_2 s_2^\pm} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{i(Q_1\theta_1+Q_2\theta_2)}{4}} \right)^{-1} \frac{(e^{-i(Q_1\theta_2+Q_2\theta_2)} q^{\frac{(4-\Delta)}{2}}; q^2)_\infty}{(e^{+i(Q_1\theta_1+Q_2\theta_2)} q^{\frac{(2+\Delta)}{2}}; q^2)_\infty} & \text{for } (-1)^{Q_1 s_1^\pm + Q_2 s_2^\pm} = -1, \end{cases} \end{aligned} \quad (2.13)$$

$$\textcircled{s_1^\pm, \theta_1} \xrightarrow[Q_1]{\textcircled{\Phi_1 \atop \Phi_2}} \textcircled{s_2, \theta_{2\pm}} = \left(q^{\frac{(1-\Delta)}{2}} e^{-iQ_1\theta_1} \right)^{|Q_2 s_2|} \frac{(e^{-i(Q_1\theta_1+Q_2\theta_{2\pm})} q^{|Q_2 s_2| + \frac{(2-\Delta)}{2}}; q)_\infty}{(e^{+i(Q_1\theta_1+Q_2\theta_{2\pm})} q^{|Q_2 s_2| + \frac{(0+\Delta)}{2}}; q)_\infty}, \quad (2.14)$$

$$\textcircled{s_1, \theta_{1\pm}} \xrightarrow[Q_1]{\textcircled{\Phi_1 \atop \Phi_2}} \textcircled{s_2, \theta_{2\pm}} = \left(q^{\frac{1-\Delta}{2}} \right)^{|(Q_1 s_1 + Q_2 s_2)|} \frac{(e^{-i(Q_1\theta_{1\pm} + Q_2\theta_{2\pm})} q^{|(Q_1 s_1 + Q_2 s_2)| + \frac{(2-\Delta)}{2}}; q)_\infty}{(e^{+i(Q_1\theta_{1\pm} + Q_2\theta_{2\pm})} q^{|(Q_1 s_1 + Q_2 s_2)| + \frac{(0+\Delta)}{2}}; q)_\infty}, \quad (2.15)$$

$$\textcircled{s^\pm, \theta} \xrightarrow{\text{BF}} \textcircled{s, \theta_\pm} = e^{is^\pm \theta_\pm} e^{is\theta}, \quad \textcircled{s^\pm, \theta} \xrightarrow{-\text{BF}} \textcircled{s, \theta_\pm} = e^{-is^\pm \theta_\pm} e^{-is\theta}. \quad (2.16)$$

We write here doubly charged matter multiplets in (2.13), (2.14) and (2.15), but if one want to consider singly charged matter, the formula is obtained just by turning off one of charges Q_1 or Q_2 . In addition, we can couple matter rectangular node to circular nodes more than three, in this case, the formula is generalized naturally just by replacing

$$Q_S, Q_{s^\pm} \rightarrow \sum_i Q_i s_i, \sum_i Q_i s_i^\pm, \quad Q_\theta, Q_{\theta_\pm} \rightarrow \sum_i Q_i \theta_i, \sum_i Q_i \theta_{i\pm},$$

in all expression.

2.2 S-operation

Interpretation for background vector multiplets. For later use, let us remind the physical meaning of background vector multiplets, e.g. (2.8) or (2.9). Basically, turning on s^\pm or s corresponds to changing the Hilbert space because turning on the flux generates Landau levels in the sense of quantum mechanics. On the other hand, turning on θ or θ_\pm is equivalent to consider twisted boundary condition:³

$$\left(\dots \longrightarrow \textcircled{s^\pm, \theta} \right) \sim \text{Tr}_{\mathcal{H}_{s^\pm}^{\mathbb{RP}^2}} \left[(-1)^{\hat{F}} q^{\frac{1}{2}(\hat{H} + \hat{j}_3)} e^{-i\theta} \right] := \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, e^{i\theta}), \quad (2.17)$$

$$\left(\dots \rightleftharpoons \textcircled{s, \theta_\pm} \right) \sim \text{Tr}_{\mathcal{H}_s^{\mathbb{RP}^2}} \left[(-1)^{\hat{F}} q^{\frac{1}{2}(\hat{H} + \hat{j}_3)} e^{-i\theta_\pm} \right] := \mathcal{I}^{\mathbb{RP}^2}(q; s, e^{i\theta_\pm}). \quad (2.18)$$

S-operation. Usually, S-operation was defined by using topological BF term as

$$\left(\dots \longrightarrow \textcircled{} \right) \xrightarrow{S^{\pm 1}} \left(\dots \longrightarrow \textcircled{} \text{---} \textcircled{\pm \text{BF}} \textcircled{s, \theta_\pm} \right). \quad (2.19)$$

However, as we emphasized, there are two types of vector multiplets in our case, and the BF term is only possible between different types of nodes (2.7). It means that the type of external vector multiplet in (2.19) should be exchanged under the S-operation:

$$\begin{aligned} &\mathcal{P}\text{-type background} \xrightarrow{S^{\pm 1}} \mathcal{CP}\text{-type background} \\ &\left(\dots \longrightarrow \textcircled{s^\pm, \theta} \right) \xrightarrow{S^{\pm 1}} \left(\dots \longrightarrow \textcircled{\mathcal{P}} \text{---} \textcircled{\pm \text{BF}} \textcircled{s, \theta_\pm} \right) \end{aligned} \quad (2.20)$$

$$\begin{aligned} &\mathcal{CP}\text{-type background} \xrightarrow{S^{\pm 1}} \mathcal{P}\text{-type background} \\ &\left(\dots \rightleftharpoons \textcircled{s, \theta_\pm} \right) \xrightarrow{S^{\pm 1}} \left(\dots \rightleftharpoons \textcircled{\mathcal{CP}} \text{---} \textcircled{\pm \text{BF}} \textcircled{s^\pm, \theta} \right) \end{aligned} \quad (2.21)$$

Then, we can observe the expected characters, $SS^{-1} = 1$, $S^2 = -1$ by utilizing the following formulas:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{is\theta} = \delta_{s,0}, \quad \sum_{s \in \mathbb{Z}} e^{is\theta} = 2\pi \delta(\theta), \quad (2.22)$$

$$\frac{1}{2} \sum_{\theta_\pm=0,\pi} e^{is^\pm \theta_\pm} = \delta_{s^\pm,0}^{\text{mod } 2}, \quad \sum_{s^\pm=0,1} e^{is^\pm \theta_\pm} = 2\delta^{\text{mod } 2}(\theta_\pm), \quad (2.23)$$

where we define mod 2 delta functions:

$$\delta_{s^\pm,0}^{\text{mod } 2} = \begin{cases} 1 & \text{for } s^\pm \equiv 0 \pmod{2} \\ 0 & \text{for } s^\pm \equiv 1 \pmod{2} \end{cases}, \quad \delta^{\text{mod } 2}(\theta_\pm) = \begin{cases} 1 & \text{for } \theta_\pm/\pi \equiv 0 \pmod{2} \\ 0 & \text{for } \theta_\pm/\pi \equiv 1 \pmod{2} \end{cases} \quad (2.24)$$

³This is true for background vector multiplet coupled to matter multiplets. For background vector multiplet coupled to dynamical vector multiplets via topological coupling, one should regard the dynamical vector multiplet as matter multiplet by using dual picture through $*F = d\rho$ where ρ is real scalar.

3 Mirror symmetry via loop operators

Once we regard our index in path integral form like in (2.3), we can insert operator \mathcal{O} like

$$\int_{\mathbb{RP}^2 \times \mathbb{S}^1} \mathcal{D}\Phi \mathcal{D}V \mathcal{O} e^{-S}. \quad (3.1)$$

In order to apply the supersymmetric localization technique to calculate it, the operator should preserve part of the supersymmetry, called BPS condition.

3.1 BPS loop operators

We can find such BPS Wilson loop which wraps \mathbb{S}^1 -direction standing at north pole = south pole of \mathbb{RP}^2 . To define Wilson loop, gauge field is necessary. In our case, vector multiplet is necessary. As we noted, there are two types of vector multiplet. We find the following operators are BPS in each type

$$\mathcal{P}\text{-type BPS Wilson loop: } W_e^{(\mathcal{P})} = e^{2ie \int_0^{2\pi} dy A_y|_{\vartheta=0,\pi}}, \quad (3.2)$$

$$\mathcal{CP}\text{-type BPS Wilson loop: } W_e^{(\mathcal{CP})} = e^{-ie \int_0^{2\pi} dy \sigma|_{\vartheta=0,\pi}}. \quad (3.3)$$

On the other hand, one can define *disorder* operator called Vortex loop operator by considering singular boundary condition along the loop. On \mathbb{S}^3 or $\mathbb{S}^2 \times \mathbb{S}^1$, these supersymmetric operators are defined in somewhat direct way [9], but one can utilize S-operation to define them as done on \mathbb{S}^3 by [8]. We have already observed the S-operation works well in our case, so we use this formula to define vortex loops for two-types:

$$\mathcal{P}\text{-type BPS Vortex loop: } V_m^{(\mathcal{P})} \xleftarrow{S} W_m^{(\mathcal{CP})}, \quad (3.4)$$

$$\mathcal{CP}\text{-type BPS Vortex loop: } V_m^{(\mathcal{CP})} \xleftarrow{S} W_m^{(\mathcal{P})}. \quad (3.5)$$

Now, from the point of view that S-operation is “particle-vortex duality”, it is understood as \mathcal{P} -type vortex = \mathcal{CP} -type particle, and \mathcal{CP} -type vortex = \mathcal{P} -type particle. See appendix A for more detail.

3.2 Loop operator effects on localization computation

First, let us define quiver notation for inserting the above loop operators. We just represent insertion of loop operator $\mathcal{O} = W_e, V_m$ associated symmetry by just adding appropriate subscript to the circular node:

$$\hat{\mathcal{O}} \cdot \mathcal{I}^{\mathbb{RP}^2}(q) := \int \dots \mathcal{D}V \mathcal{O}(V) e^{-S} = \left(\dots \longrightarrow \bigcirc_{\mathcal{O}} \right), \quad (3.6)$$

then, the recipe for the exact result is as follows:

- 1. Follow the previous recipe in (2.11)–(2.16)

- 2. For the subscript $\mathcal{O} = W_e$ or V_m , follow the rules:

$$\begin{aligned} \left(\dots \longrightarrow \textcircled{s^\pm, \theta}_{W_e} \right) &= e^{2ie\theta} \left(\dots \longrightarrow \textcircled{s^\pm, \theta} \right), \\ \left(\dots \longrightarrow \textcircled{s^\pm, \theta}_{V_m} \right) &= e^{-2\pi m \frac{\partial}{\partial \theta}} \left(\dots \longrightarrow \textcircled{s^\pm, \theta} \right), \end{aligned} \quad (3.7)$$

$$\begin{aligned} \left(\dots \longleftrightarrow \textcircled{s, \theta_\pm}_{W_e} \right) &= e^{2\pi i e s} \left(\dots \longleftrightarrow \textcircled{s, \theta_\pm} \right), \\ \left(\dots \longleftrightarrow \textcircled{s, \theta_\pm}_{V_m} \right) &= e^{-2m \frac{\partial}{\partial s}} \left(\dots \longleftrightarrow \textcircled{s, \theta_\pm} \right). \end{aligned} \quad (3.8)$$

Generalized index and its loop insertions. We can regard the above formulas in (3.7) and (3.8) in the context of generalized indices (2.17) or (2.18). In this case,

$$\text{wilson loop: } w_e, \quad \text{vortex loop: } v_m \quad (3.9)$$

are associated by the external fields, and we get

$$\hat{w}_e \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta) = e^{2ie\theta} \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta), \quad \hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta) = \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta - 2\pi m), \quad (3.10)$$

$$\hat{w}_e \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta_\pm) = e^{2\pi i e s} \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta_\pm), \quad \hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta_\pm) = \mathcal{I}^{\mathbb{RP}^2}(q; s - 2m, \theta_\pm), \quad (3.11)$$

One can find the Vortex operators act on the indices as finite differential operators. We can regard the gauging procedure as the S-transform itself, i.e.

$$\hat{W}_e = \hat{S} \hat{w}_e, \quad \hat{V}_m = \hat{S} \hat{v}_m. \quad (3.12)$$

Therefore, we get

$$\hat{W}_e = \hat{S} \hat{w}_e \hat{S}^{-1} \hat{S} = \hat{v}_{-e} \hat{S}, \quad \hat{V}_m = \hat{S} \hat{v}_m \hat{S}^{-1} \hat{S} = \hat{w}_{-m} \hat{S}. \quad (3.13)$$

Note that this equalities mean the statements in (1.4) and (1.5) because S-transformation exchanges the \mathcal{P} and \mathcal{CP} .

3.3 Mirror symmetry with loop operators

The expectation values of loop operators associated to $U(1)_A$ symmetry should be in agreement if the original indices are identical. We can find somewhat non-trivial statement in [9, 11] that

$$\bullet \text{ Gauge Wilson loop for SQED} \Leftrightarrow \text{Flavor } U(1)_V \text{ Vortex loop for XYZ.} \quad (3.14)$$

$$\bullet \text{ Gauge Vortex loop for SQED} \Leftrightarrow \text{Flavor } U(1)_V \text{ Wilson loop for XYZ.} \quad (3.15)$$

These can be easily checked in our case. Because of the algebraic equalities (3.13), we get

$$\bullet \text{ Gauge Wilson loop for SQED} = \text{Topological } U(1)_J \text{ Vortex loop for SQED.} \quad (3.16)$$

$$\bullet \text{ Gauge Vortex loop for SQED} = \text{Topological } U(1)_J \text{ Wilson loop for SQED.} \quad (3.17)$$

In addition, we already know that topological $U(1)_J$ in the SQED corresponds to flavor $U(1)_V$ in the XYZ model under the mirror map. As a consequence, the dualities of the loop operator insertions are claimed between for the gauge symmetry and for the global symmetry as (3.14) and (3.15).

4 Multi-flavor mirror symmetry

4.1 From single-flavor to multi-flavor mirror symmetry

Abelian 3d $\mathcal{N} = 2$ mirror symmetry [1–4] with a single matter is the duality between the SQED and the XYZ model, namely, they flow to the same fixed point in IR. The moduli parameters in the SQED, the combination of the scalar in the vector multiplet and the dual photon $e^{\pm(\sigma+i\rho)}$ for Coulomb branch, and the product of matters $\tilde{Q}Q$ for Higgs branch, correctly match (X, Y) and Z which are matter multiplets in the XYZ model, respectively. Furthermore, there are two global symmetries $U(1)_J$ (as a topological one) and $U(1)_A$ in addition to an R-symmetry in the SQED, and the XYZ model has exactly corresponding flavor symmetries $U(1)_V$ and $U(1)_A$ acting on the matter fields as for usual dualities. For the neat review of this simplest version of mirror symmetry and charge assignments on the fields, see [5, 7]. This symmetry can be directly checked as the equality of the superconformal indices computed by the localization [7, 13],

$$\mathcal{I}_{\text{SQED}}^{\mathcal{N}=2} = \mathcal{I}_{\text{XYZ}}^{\mathcal{N}=2}, \quad (4.1)$$

for the SQED $\mathcal{I}_{\text{SQED}}^{\mathcal{N}=2}$ and the XYZ model $\mathcal{I}_{\text{XYZ}}^{\mathcal{N}=2}$ (from now on, we omit the superscript \mathbb{RP}^2 on the index).

We can enlarge supersymmetry to $\mathcal{N} = 4$ [1–3] by adding an adjoint chiral multiplet \tilde{S} to the $\mathcal{N} = 2$ SQED in such a way that it couples with the matters through a superpotential $\tilde{Q}\tilde{S}Q$. Correspondingly, in the XYZ model, we need to couple a chiral multiplet \tilde{Z} to Z , which is equivalent to introducing the mass term $\tilde{Z}Z$ for them. Thus, they are integrated out, and the resultant matter theory becomes the free theory only with X and Y . In the language of the index which is known to be a protected quantity along the RG flow, those processes can be done simply by multiplying the inverse of the one-loop determinant $\mathcal{Z}_{1\text{-loop}}^Z$ for Z , or equivalently, the one-loop determinant $\mathcal{Z}_{1\text{-loop}}^{\tilde{Z}}$ for \tilde{Z} , by both sides of the equality (4.1). This is just the observation that integrating out Z in the XYZ model is to simply remove its one-loop determinant from the index, and, on the other hand, the new adjoint field \tilde{S} in the SQED has charges under the global symmetries corresponding to these of \tilde{Z} which can be read off from the superpotential. Thus, the contribution $\mathcal{Z}_{1\text{-loop}}^{\tilde{S}}$ for \tilde{S} to the index can be viewed as the one $\mathcal{Z}_{1\text{-loop}}^{\tilde{Z}}$ for \tilde{Z} under mirror symmetry, and then schematically, $\mathcal{N} = 4$ mirror symmetry should be written as

$$\mathcal{I}_{\text{SQED}}^{\mathcal{N}=4} = \mathcal{I}_{\text{free}}^{\mathcal{N}=4} \quad \Leftrightarrow \quad \mathcal{Z}_{1\text{-loop}}^{\tilde{S}} \mathcal{I}_{\text{SQED}}^{\mathcal{N}=2} = \mathcal{Z}_{1\text{-loop}}^{\tilde{Z}} \mathcal{I}_{\text{XYZ}}^{\mathcal{N}=2}, \quad (4.2)$$

where $\mathcal{I}_{\text{SQED}}^{\mathcal{N}=4}$ and $\mathcal{I}_{\text{free}}^{\mathcal{N}=4}$ are the indices for the $\mathcal{N} = 4$ SQED and for the free matter theory, respectively, and

$$\mathcal{Z}_{1\text{-loop}}^{\tilde{S}} = \mathcal{Z}_{1\text{-loop}}^{\tilde{Z}} = (\mathcal{Z}_{1\text{-loop}}^Z)^{-1} \quad (4.3)$$

under the suitable identification of parameters. The index formula (4.2) of $\mathcal{N} = 4$ mirror symmetry with a single flavor is a key to demonstrate $\mathcal{N} = 2$ mirror symmetry with many flavors. We will show these explicit forms depending on the parity conditions in the next subsection.

Now, we aim to briefly review the statement of $\mathcal{N} = 2$ multi-flavor mirror symmetry. The one side of the duality is the SQED with N_f a general number of matters. The dual side is the quiver theory with an $U(1)^{N_f-1}$ gauge symmetry, and there are an adjoint matter associated with each gauge node, bifundamental matters connecting two neighboring gauge nodes, and fundamental matters at the ends of the quiver diagram. The former can be obtained from the theory with N_f free matters by gauging the sum of N_f $U(1)_V$'s equipped with each single-flavor free theory. To construct the correct dual theory, we start with N_f sets of the $\mathcal{N} = 4$ SQED comprised of the $\mathcal{N} = 2$ SQED and an adjoint matter coupled to the fundamental matter via the superpotential, then gauging the sum of N_f $U(1)_J$'s. Note that this gauging process is equivalent to ungauging the diagonal $U(1)$ part of the $U(1)^{N_f}$ gauge symmetry, which results in the $U(1)^{N_f-1}$ quiver gauge theory with appropriately redefining the gauge charges of the bifundamental fields. Actually, the above theories which we use as starting points to get multi-flavor mirror symmetry are exactly connected to each other by $\mathcal{N} = 4$ mirror symmetry: as explained, the $N_f = 1$ free matter theory is dual to the $\mathcal{N} = 4$ SQED with a single flavor. Therefore, we take the following strategy [7] based on that fact to build multi-flavor mirror symmetry in terms of the index:

1. We set the index \mathcal{I}_{N_f} as the collection of N_f free single-matter theories, that is, $\mathcal{I}_{N_f} = \prod_{i=1}^{N_f} \mathcal{I}_{\text{free}}^{\mathcal{N}=4}(a_i)$ where a_i is some parameter of the i -th free theory. This clearly enjoys $\mathcal{N} = 4$ mirror symmetry as

$$\mathcal{I}_{N_f} = \mathcal{I}_{U(1)^{N_f}}, \quad (4.4)$$

where $\mathcal{I}_{U(1)^{N_f}}$ is the index for N_f copies of the $\mathcal{N} = 4$ SQED with one flavor, $\mathcal{I}_{U(1)^{N_f}} = \prod_{i=1}^{N_f} \mathcal{I}_{\text{SQED}}^{\mathcal{N}=4}(a_i)$.

2. For \mathcal{I}_{N_f} in (4.4) which contains global symmetries $U(1)_V^{N_f}$ and $U(1)_A^{N_f}$, we gauge the diagonal part of $U(1)_V^{N_f}$ by taking the suitable combinations of parameters and then obtain the index $\mathcal{I}_{\text{SQED}_{N_f}}$ for the SQED with general N_f flavors.
3. For $\mathcal{I}_{U(1)^{N_f}}$ in (4.4) which contains global symmetries $U(1)_J^{N_f}$ and $U(1)_A^{N_f}$, we gauge the diagonal part of $U(1)_J^{N_f}$ by taking the suitable combinations of parameters and adjust gauge parameters to show the correct charges of the bifundamental matters. As a result, we get the index $\mathcal{I}_{U(1)^{N_f-1}}$ for the $U(1)^{N_f-1}$ quiver gauge theory.
4. Finally, by construction, we conclude multi-flavor mirror symmetry as the index equality

$$\mathcal{I}_{\text{SQED}_{N_f}} = \mathcal{I}_{U(1)^{N_f-1}}. \quad (4.5)$$

In the following, we say a \mathcal{P} -type or a \mathcal{CP} -type $U(1)$ symmetry for short if the background gauge field for that $U(1)$ respects the \mathcal{P} -type or the \mathcal{CP} -type parity condition, respectively.

4.2 Single-flavor $\mathcal{N} = 4$ mirror symmetry

In this subsection, we will give the formulas for $\mathcal{N} = 4$ mirror symmetry which play a central role to derive the multi-flavor version of $\mathcal{N} = 2$ mirror symmetry. As explained, they can be readily derived from $\mathcal{N} = 2$ mirror symmetry whose detailed discussions including the charge assignment under the symmetries and rigorous proof can be seen in [5, 6].

There are two types of Abelian mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$ depending on the case where the dynamical vector multiplet in the SQED is set to be \mathcal{P} -type or \mathcal{CP} -type. We name the former \mathcal{P} -SQED $^{(\pm)}$ and the latter \mathcal{CP} -SQED $^{(\pm)}$. The superscript (\pm) represents a fixed value of the background field coupled to the dynamical one though a BF term.

$\mathcal{N} = 4$ \mathcal{P} -SQED $^{(\pm)}$ vs the free matter theory.

$$\begin{aligned} \mathcal{I}_{\mathcal{P}\text{-SQED}^{(\pm)}}^{\mathcal{N}=4}(q; a; s, e^{i\theta_{J\pm}}) & \\ = a^{\frac{1}{4}} \frac{(q^2, a^{-1}q; q^2)_\infty}{(q, a; q^2)_\infty} \sum_{s^\pm} \oint \frac{dz}{2\pi iz} e^{is^\pm \theta_{J\pm} z^s} \left\{ a^{-\frac{1}{4} + \frac{s^\pm}{2}} \frac{(z^{-1} a^{\frac{1}{2}} q^{\frac{1}{2} + s^\pm}, z a^{\frac{1}{2}} q^{\frac{1}{2} + s^\pm}; q^2)_\infty}{(z a^{-\frac{1}{2}} q^{\frac{1}{2} + s^\pm}, z^{-1} a^{-\frac{1}{2}} q^{\frac{1}{2} + s^\pm}; q^2)_\infty} \right\}, \end{aligned} \quad (4.6)$$

$$\mathcal{I}_{\text{free}}^{\mathcal{N}=4}(q; \tilde{a}; \tilde{s}, e^{i\theta_{V\pm}}) = \left(\tilde{a}^{-\frac{1}{2}} q^{\frac{1}{2}} \right)^{|\tilde{s}|} \frac{(e^{i\theta_{V\pm}} \tilde{a}^{-\frac{1}{2}} q^{1+|\tilde{s}|}; q)_\infty}{(e^{i\theta_{V\pm}} \tilde{a}^{\frac{1}{2}} q^{|\tilde{s}|}; q)_\infty}. \quad (4.7)$$

The index formula for $\mathcal{N} = 4$ \mathcal{P} -SQED $^{(\pm)}$ is given by (4.6). The parameter $a := e^{-2i\theta_A} q^{(1-\Delta)}$ includes a Wilson line phase θ_A associated with an axial $U(1)_A$ global symmetry, and $(s, \theta_{J\pm})$ are a flux and a discretized Wilson line for \mathcal{CP} -type $U(1)_J$. The leftmost q -Pochhammer symbol of (4.6) corresponds to the contribution for the adjoint matter with charge -2 under $U(1)_A$ in addition to the $U(1)$ vector multiplet. (4.7) is the index for the dual free theory with $\tilde{a} := e^{+2i\tilde{\theta}_A} q^{(1-\Delta)}$ where $\tilde{\theta}_A$ is a Wilson line for $U(1)_A$, and $(\tilde{s}, \theta_{V\pm})$ are a flux and a discretized Wilson line for \mathcal{CP} -type $U(1)_V$ in this theory. Single-flavor $\mathcal{N} = 4$ mirror symmetry is indicated by the equality of these indices (4.6) and (4.7) under the map $a = \tilde{a}$ and $(s, \theta_{J\pm}) = (\tilde{s}, \theta_{V\pm})$.

$\mathcal{N} = 4$ \mathcal{CP} -SQED $^{(\pm)}$ vs the free matter theory.

$$\begin{aligned} \mathcal{I}_{\mathcal{CP}\text{-SQED}^{(\pm)}}^{\mathcal{N}=4}(q; a; s^\pm, w) & \\ = a^{-\frac{1}{4}} \frac{(q, a; q^2)_\infty}{(q^2, a^{-1}q; q^2)_\infty} \sum_{s \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_\pm} e^{is^\pm \theta_\pm} w^s \left(q^{\frac{1}{2}} a^{-\frac{1}{2}} \right)^{|s|} \left[\frac{(e^{i\theta_\pm} a^{-\frac{1}{2}} q^{|s|+1}; q)_\infty}{(e^{i\theta_\pm} a^{\frac{1}{2}} q^{|s|}; q)_\infty} \right], \end{aligned} \quad (4.8)$$

$$\mathcal{I}_{\text{free}}^{\mathcal{N}=4}(q; \tilde{a}; \tilde{s}^\pm, \tilde{w}) = \tilde{a}^{-\frac{1}{4} + \frac{\tilde{s}^\pm}{2}} \frac{(\tilde{a}^{\frac{1}{2}} \tilde{w}^{-1} q^{\frac{1}{2} + \tilde{s}^\pm}, \tilde{a}^{\frac{1}{2}} \tilde{w} q^{\frac{1}{2} + \tilde{s}^\pm}; q^2)_\infty}{(\tilde{a}^{-\frac{1}{2}} \tilde{w} q^{\frac{1}{2} + \tilde{s}^\pm}, \tilde{a}^{-\frac{1}{2}} \tilde{w}^{-1} q^{\frac{1}{2} + \tilde{s}^\pm}; q^2)_\infty}. \quad (4.9)$$

The other version of $\mathcal{N} = 4$ mirror symmetry is described by the indices (4.8) for $\mathcal{N} = 4$ \mathcal{CP} -SQED $^{(\pm)}$ and (4.9) for the dual free theory. In the former, the parameter s is a holonomy and w is a fugacity for \mathcal{P} -type $U(1)_J$, on the other hand, $(\tilde{s}^\pm, \tilde{w})$ are a holonomy and a fugacity for \mathcal{P} -type $U(1)_V$ in the dual theory. $\mathcal{N} = 4$ mirror symmetry with one flavor is realized as the equality of (4.8) and (4.9) under identifications $a = \tilde{a}$ and $(s^\pm, w) = (\tilde{s}^\pm, \tilde{w})$. The mathematical proof to show equivalence of the indices in both cases is done in [5, 6].

4.3 Multi-flavor $\mathcal{N} = 2$ mirror symmetry

We will demonstrate systematically the multi-flavor version of Abelian mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$ following the prescription explained in section 4.1. The starting point is the index \mathcal{I}_{N_f} of N_f free matter multiplets coupled to the background gauge fields associated with N_f $U(1)_V$ flavor symmetries which rotate a phase to each matter multiplet. We should notice that even though *a priori* each background field can take freely a \mathcal{P} - or a \mathcal{CP} -type parity condition, it seems to be only allowed to set all of parity conditions being *either* \mathcal{P} - or \mathcal{CP} -type because we may gauge the diagonal part of $U(1)_V^{N_f}$ if all of them are the same. It is apparently unable to construct multi-flavor mirror symmetry with the combination of an arbitrary number of \mathcal{P} - and \mathcal{CP} -type $U(1)_V$'s. This is why, in this paper, we have two kinds of the SQEDs in the context of multi-flavor mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$, that is, \mathcal{P} -SQED $_{N_f}$ which is the theory obtained by gauging the sum of \mathcal{P} -type $U(1)_V^{N_f}$ in the N_f -free matter theory and \mathcal{CP} -SQED $_{N_f}$ by gauging the sum of \mathcal{CP} -type $U(1)_V^{N_f}$.

\mathcal{P} -SQED $_{N_f}$ vs \mathcal{CP} - $U(1)^{N_f-1}$ gauge theory. We first consider \mathcal{P} -SQED $_{N_f}$ and its dual theory. We begin with the index $\mathcal{I}_{N_f}^{(\mathcal{P})}$ for the free theory with N_f \mathcal{P} -type $U(1)_V$'s given by

$$\mathcal{I}_{N_f}^{(\mathcal{P})}(q; \{a_i\}; \{s_i^\pm\}, \{w_i\}) = \prod_{i=1}^{N_f} a_i^{-\frac{1}{4} + \frac{1}{2}s_i^\pm} \frac{(a_i^{\frac{1}{2}} w_i^{-1} q^{\frac{1}{2} + s_i^\pm}, a_i^{\frac{1}{2}} w_i q^{\frac{1}{2} + s_i^\pm}; q^2)_\infty}{(a_i^{-\frac{1}{2}} w_i q^{\frac{1}{2} + s_i^\pm}, a_i^{-\frac{1}{2}} w_i^{-1} q^{\frac{1}{2} + s_i^\pm}; q^2)_\infty}, \quad (4.10)$$

where s_i^\pm corresponds to a holonomy on \mathbb{RP}^2 and w_i is a fugacity associated to each $U(1)_V$. A fugacity a_i contains a parameter associated to each \mathcal{P} -type $U(1)_A$ symmetry as defined in single-flavor $\mathcal{N} = 4$ mirror symmetry. We rewrite this index (4.10) as follows:

- \mathcal{P} -SQED $_{N_f}$:

to extract the diagonal part of $U(1)_V^{N_f}$, we here redefine the parameters of $\mathcal{I}_{N_f}^{(\mathcal{P})}$ as

$$\begin{aligned} s_i^\pm &= m_i^\pm + m^\pm \quad \text{with} \quad \sum_i m_i^\pm \equiv 0 \pmod{2}, \\ w_i &= z u_i \quad \text{with} \quad \prod_i u_i = 1. \end{aligned} \quad (4.11)$$

The parameters (m^\pm, z) are ones for the diagonal part of $U(1)_V^{N_f}$ which we will denote by $U(1)_{\text{diag}}$. Gauging $U(1)_{\text{diag}}$ in this free theory make it \mathcal{P} -SQED $_{N_f}$ whose index is

$$\begin{aligned} &\mathcal{I}_{\mathcal{P}\text{-SQED}_{N_f}}(q; \{a_i\}; \{m_i^\pm\}, \{u_i\}; m_J, \theta_{J\pm}) \\ &= q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{m^\pm} \oint \frac{dz}{2\pi i z} e^{im^\pm \theta_{J\pm}} z^{m_J} \\ &\quad \times \prod_{i=1}^{N_f} a_i^{-\frac{1}{4} + \frac{1}{2}(m_i^\pm + m^\pm)} \frac{(a_i^{\frac{1}{2}}(z u_i)^{-1} q^{\frac{1}{2} + m_i^\pm + m^\pm}, a_i^{\frac{1}{2}}(z u_i) q^{\frac{1}{2} + m_i^\pm + m^\pm}; q^2)_\infty}{(a_i^{-\frac{1}{2}}(z u_i) q^{\frac{1}{2} + m_i^\pm + m^\pm}, a_i^{-\frac{1}{2}}(z u_i)^{-1} q^{\frac{1}{2} + m_i^\pm + m^\pm}; q^2)_\infty}, \end{aligned} \quad (4.12)$$

where we introduce parameters $m_J \in \mathbb{Z}$ and $\theta_{J\pm}$ for a \mathcal{CP} -type $U(1)_J$ topological symmetry from gauging $U(1)_{\text{diag}}$ because a background field for $U(1)_J$ should be coupled to a \mathcal{P} -type dynamical one via the BF term. This is the index for \mathcal{P} -SQED $_{N_f}$.

- $\mathcal{CP}\text{-U}(1)^{N_f-1}$ gauge theory:

we refer to the theory dual to $\mathcal{P}\text{-SQED}_{N_f}$ as $\mathcal{CP}\text{-U}(1)^{N_f-1}$ which is comprised of $(N_f - 1)$ \mathcal{CP} -type $\text{U}(1)$ gauge groups with global symmetries corresponding to these in $\mathcal{P}\text{-SQED}_{N_f}$. Returning back to $\mathcal{I}_{N_f}^{(\mathcal{P})}$ (4.10), we apply the identity between (4.6) and (4.7) representing $\mathcal{N} = 4$ mirror symmetry to re-express $\mathcal{I}_{N_f}^{(\mathcal{P})}$ as an equivalent but quite different form $\mathcal{I}_{\mathcal{CP}\text{-U}(1)^{N_f}}$,

$$\begin{aligned} & \mathcal{I}_{\mathcal{CP}\text{-U}(1)^{N_f}}(q; \{a_i\}; \{s_i^\pm\}, \{w_i\}) \\ &= \prod_{i=1}^{N_f} a_i^{-\frac{1}{4}} \frac{(q, a_i; q^2)_\infty}{(q^2, a_i^{-1}q; q^2)_\infty} \sum_{s_i \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_{i\pm}} e^{is_i^\pm \theta_{i\pm}} w_i^{s_i} \left(q^{\frac{1}{2}} a_i^{-\frac{1}{2}} \right)^{|s_i|} \frac{(e^{i\theta_{i\pm}} a_i^{-\frac{1}{2}} q^{|s_i|+1}; q)_\infty}{(e^{i\theta_{i\pm}} a_i^{\frac{1}{2}} q^{|s_i|}; q)_\infty}. \end{aligned} \quad (4.13)$$

This is simply the index for N_f copies of $\mathcal{CP}\text{-SQED}^{(\pm)}$'s having N_f adjoint matters and N_f \mathcal{P} -type $\text{U}(1)_J$ global symmetries whose labels are (s_i^\pm, w_i) . Here, we set new variables (m_i^\pm, u_i) for these global symmetries such as

$$s_i^\pm = m_i^\pm + m^\pm, \quad w_i = zu_i, \quad (4.14)$$

where (m^\pm, z) correspond to a holonomy and a Wilson line phase for the sum of N_f \mathcal{P} -type $\text{U}(1)_J$'s which will be denoted by $\widetilde{\text{U}(1)}_{\text{diag}}$. Then, by gauging this \mathcal{P} -type $\widetilde{\text{U}(1)}_{\text{diag}}$, we acquire the index for $\mathcal{CP}\text{-U}(1)^{N_f-1}$ as

$$\begin{aligned} & \mathcal{I}_{\mathcal{CP}\text{-U}(1)^{N_f-1}}(q; \{a_i\}; \{m_i^\pm\}, \{u_i\}; m_J, \theta_{J\pm}) \\ &= q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{m^\pm} \oint \frac{dz}{2\pi iz} e^{im^\pm \theta_{J\pm}} z^{m_J} \\ &\quad \times \prod_{i=1}^{N_f} a_i^{-\frac{1}{4}} \frac{(q, a_i; q^2)_\infty}{(q^2, a_i^{-1}q; q^2)_\infty} \sum_{s_i \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_{i\pm}} e^{i(m_i^\pm + m^\pm) \theta_{i\pm}} (zu_i)^{s_i} \left(q^{\frac{1}{2}} a_i^{-\frac{1}{2}} \right)^{|s_i|} \frac{(e^{i\theta_{i\pm}} a_i^{-\frac{1}{2}} q^{|s_i|+1}; q)_\infty}{(e^{i\theta_{i\pm}} a_i^{\frac{1}{2}} q^{|s_i|}; q)_\infty}, \end{aligned} \quad (4.15)$$

where $m_J \in \mathbb{Z}$ and $\theta_{J\pm}$ are a monopole flux and a Wilson line phase, respectively, for a topological $\widetilde{\text{U}(1)}_J$ which emerges from gauging $\widetilde{\text{U}(1)}_{\text{diag}}$. Now, we can evaluate the summations over m^\pm and z . The sum over m^\pm gives the delta function (2.22) defined by mod 2

$$\sum_{m^\pm} e^{im^\pm(\theta_{J\pm} + \sum_i \theta_{i\pm})} = 2\delta^{\text{mod } 2}(\theta_{J\pm} + \sum_i \theta_{i\pm}), \quad (4.16)$$

and the integral over z reduces to the ordinary Kronecker delta

$$\oint \frac{dz}{2\pi iz} z^{m_J + \sum_i s_i} = \delta_{m_J + \sum_i s_i, 0}. \quad (4.17)$$

By utilizing these constraints, we can split $(\theta_{i\pm}, s_i)$ into other (hatted) variables as follows:

$$\begin{aligned} (4.16) &\Rightarrow \theta_{i\pm} = \hat{\theta}_{i\pm} - \hat{\theta}_{i+1,\pm} - \frac{1}{N_f} \theta_{J\pm} \pmod{2\pi}, \\ (4.17) &\Rightarrow s_i = \hat{s}_i - \hat{s}_{i+1} - \frac{1}{N_f} m_J. \end{aligned} \quad (4.18)$$

	$\mathcal{P}\text{-SQED}_{N_f}$	$\mathcal{CP}\text{-U}(1)^{N_f-1}$
Gauge	$\mathcal{P}\text{-type U}(1)_{\text{diag}} : (m^\pm, z)$	$\mathcal{CP}\text{-type U}(1)^{N_f-1} : (\hat{s}_i, \hat{\theta}_{i\pm})$
Global	$\mathcal{CP}\text{-type U}(1)_J : (m_J, \theta_{J\pm})$	$\mathcal{CP}\text{-type } \widetilde{\text{U}(1)}_J : (m_J, \theta_{J\pm})$
	$\mathcal{P}\text{-type U}(1)_V^{N_f}/\text{U}(1)_{\text{diag}} : (m_i^\pm, u_i)$	$\mathcal{P}\text{-type U}(1)_J^{N_f}/\widetilde{\text{U}(1)}_{\text{diag}} : (m_i^\pm, u_i)$
	$\mathcal{P}\text{-type U}(1)_A^{N_f} : a_i$	$\mathcal{P}\text{-type U}(1)_A^{N_f} : a_i$

Table 1. The symmetries and their fugacities in $\mathcal{P}\text{-SQED}_{N_f}$ and $\mathcal{CP}\text{-U}(1)^{N_f-1}$.

Note that m_J should be a multiple of N_f due to gauge invariance so that \hat{s}_i are still integers. Then, the index for the dual theory $\mathcal{CP}\text{-U}(1)^{N_f-1}$ can be rewritten as

$$\begin{aligned}
 & \mathcal{I}_{\mathcal{CP}\text{-U}(1)^{N_f-1}}(q; \{a_i\}; \{m_i^\pm\}, \{u_i\}; m_J, \theta_{J\pm}) \\
 &= q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \prod_{i=1}^{N_f} a_i^{-\frac{1}{4}} \frac{(q, a_i; q^2)_\infty}{(q^2, a_i^{-1}q; q^2)_\infty} \sum_{\hat{s}_i \in \mathbb{Z}} \frac{1}{2} \sum_{\hat{\theta}_{i\pm}} e^{im_i^\pm(\hat{\theta}_{i\pm} - \hat{\theta}_{i+1, \pm} - \frac{1}{N_f}\theta_{J\pm})} u_i^{\hat{s}_i - \hat{s}_{i+1} - \frac{1}{N_f}m_J} \\
 & \times \left(q^{\frac{1}{2}} a_i^{-\frac{1}{2}} \right)^{|\hat{s}_i - \hat{s}_{i+1} - \frac{1}{N_f}m_J|} \frac{(e^{i\hat{\theta}_{i\pm} - i\hat{\theta}_{i+1, \pm} - \frac{i}{N_f}\theta_{J\pm} - \frac{1}{2}|\hat{s}_i - \hat{s}_{i+1} - \frac{1}{N_f}m_J| + 1}; q)_\infty}{(e^{i\hat{\theta}_{i\pm} - i\hat{\theta}_{i+1, \pm} - \frac{i}{N_f}\theta_{J\pm} - \frac{1}{2}|\hat{s}_i - \hat{s}_{i+1} - \frac{1}{N_f}m_J|}; q)_\infty}.
 \end{aligned} \tag{4.19}$$

This is the precise expression⁴ of the index for $\mathcal{CP}\text{-U}(1)^{N_f-1}$ with showing the contribution of each bifundamental matter Φ_i ($i = 1, 2, \dots, N_f - 1$) whose charge is $(1, -1)$ only under $(\text{U}(1)_i, \text{U}(1)_{i+1})$ gauge groups in the quiver diagram.

The equality of the indices (4.12) for $\mathcal{P}\text{-SQED}_{N_f}$ and (4.19) for $\mathcal{CP}\text{-U}(1)^{N_f-1}$ gauge theory is just Abelian $\mathcal{N} = 2$ mirror symmetry with general N_f flavors. We summarize the symmetries and its fugacities contained in both theories in table 1. Although all of the global symmetries are mapped in the one-to-one manner between two theories, we find that types of the parity conditions for the gauge groups *are exchanged* with each other. Surely, the gauge symmetries in several theories which share the common IR fixed point do not necessarily match at UV, but this observation is an interesting, specific feature reflecting the fact that the theories are put on an unorientable manifold.

$\mathcal{CP}\text{-SQED}_{N_f}$ vs $\mathcal{P}\text{-U}(1)^{N_f-1}$ gauge theory. Let us turn to multi-flavor mirror symmetry between the SQED_{N_f} of the $\mathcal{CP}\text{-type}$ gauge symmetry and the dual quiver theory. The index $\mathcal{I}_{N_f}^{(\mathcal{CP})}$ of N_f free matters coupled to N_f $\mathcal{CP}\text{-type U}(1)_V$'s is written by

$$\mathcal{I}_{N_f}^{(\mathcal{CP})}(q; \{a_i\}; \{s_i\}, \{\theta_{i\pm}\}) = \prod_{i=1}^{N_f} \left(a_i^{-\frac{1}{2}} q^{\frac{1}{2}} \right)^{|s_i|} \frac{(e^{i\theta_{i\pm}} a_i^{-\frac{1}{2}} q^{1+|s_i|}; q)_\infty}{(e^{i\theta_{i\pm}} a_i^{\frac{1}{2}} q^{|s_i|}; q)_\infty}, \tag{4.20}$$

where s_i is a magnetic flux and $\theta_{i\pm}$ is a Wilson line phase associated to each $\mathcal{CP}\text{-type U}(1)_V$.

⁴There is an additional shift symmetry acting as $\hat{s}_i \rightarrow \hat{s}_i + (\text{const.})$ for all i at the same time, and we can fix one of \hat{s}_i 's by this symmetry.

• \mathcal{CP} -SQED $_{N_f}$:

as done in the previous example, in order to describe \mathcal{CP} -SQED $_{N_f}$ starting from (4.20), we gauge the sum of $U(1)_{V_f}^{N_f}$ which we will name again $U(1)_{\text{diag}}$ by introducing new variables

$$\begin{aligned} s_i &= m_i + m & \text{with } \sum_i m_i &= 0, \\ \theta_{i\pm} &= \sigma_{i\pm} + \sigma_{\pm} & \text{with } \sum_i \sigma_{i\pm} &\equiv 0 \pmod{2\pi}, \end{aligned} \quad (4.21)$$

where (m, σ_{\pm}) are a monopole flux and a discretized Wilson line for \mathcal{CP} -type $U(1)_{\text{diag}}$. Thus, we obtain the index $\mathcal{I}_{\mathcal{CP}\text{-SQED}_{N_f}}$ for \mathcal{CP} -SQED $_{N_f}$ with summing over both m and σ_{\pm}

$$\begin{aligned} &\mathcal{I}_{\mathcal{CP}\text{-SQED}_{N_f}}(q; \{a_i\}; \{m_i\}, \{\sigma_i\}; m_J^{\pm}, w) \\ &= q^{-\frac{1}{8}} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{m \in \mathbb{Z}} \frac{1}{2} \sum_{\sigma_{\pm}} e^{im_J^{\pm} \sigma_{\pm}} w^m \prod_{i=1}^{N_f} \left(a_i^{-\frac{1}{2}} q^{\frac{1}{2}} \right)^{|m_i+m|} \frac{(e^{i(\sigma_{i\pm}+\sigma_{\pm})} a_i^{-\frac{1}{2}} q^{1+|m_i+m|}; q)_{\infty}}{(e^{i(\sigma_{i\pm}+\sigma_{\pm})} a_i^{\frac{1}{2}} q^{|m_i+m|}; q)_{\infty}}, \end{aligned} \quad (4.22)$$

where m_J^{\pm} and w are a holonomy and a fugacity, respectively, associated to a topological $U(1)_J$ coupling to the gauged $U(1)_{\text{diag}}$ via the BF term.

• \mathcal{P} - $U(1)^{N_f-1}$ gauge theory:

let us call \mathcal{P} - $U(1)^{N_f-1}$ the quiver theory of $(N_f - 1)$ \mathcal{P} -type $U(1)$ gauge groups with appropriate global symmetries. We can apply equivalence between (4.8) and (4.9) to rewriting the free matter index (4.20), then the resultant index $\mathcal{I}_{\mathcal{P}\text{-}U(1)^{N_f}}$ is

$$\begin{aligned} &\mathcal{I}_{\mathcal{P}\text{-}U(1)^{N_f}}(q; \{a_i\}; \{s_i\}, \{\theta_{i\pm}\}) \\ &= \prod_{i=1}^{N_f} a_i^{\frac{1}{4}} \frac{(q^2, a_i^{-1}q; q^2)_{\infty}}{(q, a_i; q^2)_{\infty}} \sum_{s_i^{\pm}} \oint \frac{dz_i}{2\pi i z_i} e^{s_i^{\pm} \theta_{i\pm}} z_i^{s_i} \left\{ a_i^{-\frac{1}{4} + \frac{1}{2}s_i^{\pm}} \frac{(z_i^{-1} a_i^{\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}, z_i a_i^{\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}; q^2)_{\infty}}{(z_i a_i^{-\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}, z_i^{-1} a_i^{-\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}; q^2)_{\infty}} \right\}. \end{aligned} \quad (4.23)$$

This shows the contributions of N_f copies of $\mathcal{N} = 4$ \mathcal{P} -SQED $^{(\pm)}$. To repeat the argument mentioned in section 4.1, we change the variables $(s_i, \theta_{i\pm})$ such as

$$s_i = m_i + m, \quad \theta_{i\pm} = \sigma_{i\pm} + \sigma_{\pm}, \quad (4.24)$$

where (m, σ_{\pm}) correspond to saddle values for the sum of $U(1)_J^{N_f}$ which we will call $\widetilde{U(1)}_{\text{diag}}$. Then, gauging $\widetilde{U(1)}_{\text{diag}}$ produces the index $\mathcal{I}_{\mathcal{P}\text{-}U(1)^{N_f-1}}$ of this dual theory

$$\begin{aligned} &\mathcal{I}_{\mathcal{P}\text{-}U(1)^{N_f-1}}(q; \{a_i\}; \{m_i\}, \{\sigma_{i\pm}\}; m_J^{\pm}, \theta_J) \\ &= q^{-\frac{1}{8}} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{m \in \mathbb{Z}} \frac{1}{2} \sum_{\sigma_{\pm}} e^{im_J^{\pm} \sigma_{\pm}} w^m \prod_{i=1}^{N_f} a_i^{\frac{1}{4}} \frac{(q^2, a_i^{-1}q; q^2)_{\infty}}{(q, a_i; q^2)_{\infty}} \\ &\quad \times \sum_{s_i^{\pm}} \oint \frac{dz_i}{2\pi i z_i} e^{is_i^{\pm}(\sigma_{i\pm}+\sigma_{\pm})} z_i^{m_i+m} \left\{ a_i^{-\frac{1}{4} + \frac{1}{2}s_i^{\pm}} \frac{(z_i^{-1} a_i^{\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}, z_i a_i^{\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}; q^2)_{\infty}}{(z_i a_i^{-\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}, z_i^{-1} a_i^{-\frac{1}{2}} q^{\frac{1}{2}+s_i^{\pm}}; q^2)_{\infty}} \right\}, \end{aligned} \quad (4.25)$$

	$\mathcal{CP}\text{-SQED}_{N_f}$	$\mathcal{P}\text{-U}(1)^{N_f-1}$
Gauge	$\mathcal{CP}\text{-type } \text{U}(1)_{\text{diag}} : (m, \sigma_{\pm})$	$\mathcal{P}\text{-type } \text{U}(1)^{N_f-1} : (\hat{s}_i^{\pm}, \hat{z}_i)$
Global	$\mathcal{P}\text{-type } \text{U}(1)_J : (m_J^{\pm}, w)$	$\mathcal{P}\text{-type } \widetilde{\text{U}(1)}_J : (m_J^{\pm}, w)$
	$\mathcal{CP}\text{-type } \text{U}(1)_V^{N_f}/\text{U}(1)_{\text{diag}} : (m_i, \sigma_{i\pm})$	$\mathcal{CP}\text{-type } \text{U}(1)_J^{N_f}/\widetilde{\text{U}(1)}_{\text{diag}} : (m_i, \sigma_{i\pm})$
	$\mathcal{P}\text{-type } \text{U}(1)_A^{N_f} : a_i$	$\mathcal{P}\text{-type } \text{U}(1)_A^{N_f} : a_i$

Table 2. The symmetries and thier fugacities in $\mathcal{CP}\text{-SQED}_{N_f}$ and $\mathcal{P}\text{-U}(1)^{N_f-1}$.

where m_J^{\pm} and w also are a holonomy and a fugacity associated to $\text{U}(1)_J$ originated from gauging $\widetilde{\text{U}(1)}_{\text{diag}}$. Now, we focus on the consequences of gauging the parameters m and σ_{\pm} . The summation over m results in an usual delta function

$$\sum_{m \in \mathbb{Z}} (w \prod_i z_i)^m = \delta(w \prod_i z_i), \quad (4.26)$$

and the summation over σ_{\pm} gives a mod 2 delta function (2.23)

$$\frac{1}{2} \sum_{\sigma_{\pm}} e^{i(m_J^{\pm} + \sum_i s_i^{\pm})\sigma_{\pm}} = \delta_{m_J^{\pm} + \sum_i s_i^{\pm}, 0}^{\text{mod } 2}. \quad (4.27)$$

As demonstrated in the previous case, to obtain the standard description of the dual theory to $\mathcal{CP}\text{-SQED}_{N_f}$, we redefine the variables by using above constraints as

$$\begin{aligned} (4.26) &\Rightarrow z_i = \hat{z}_i \hat{z}_{i+1}^{-1} w^{-\frac{1}{N_f}}, \\ (4.27) &\Rightarrow s_i^{\pm} = \hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm} \pmod{2}. \end{aligned} \quad (4.28)$$

Accordingly, the index for $\mathcal{P}\text{-U}(1)^{N_f-1}$ is written by

$$\begin{aligned} &\mathcal{I}_{\mathcal{P}\text{-U}(1)^{N_f-1}}(q; \{a_i\}; \{m_i\}, \{\sigma_{i\pm}\}; m_J^{\pm}, \theta_J) \\ &= q^{-\frac{1}{8}} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \prod_{i=1}^{N_f} a_i^{\frac{1}{4}} \frac{(q^2, a_i^{-1} q; q^2)_{\infty}}{(q, a_i; q^2)_{\infty}} \\ &\quad \times \sum_{\hat{s}_i^{\pm}} \oint \frac{d\hat{z}_i}{2\pi i \hat{z}_i} e^{i(\hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm})\sigma_{i\pm}} \left(\hat{z}_i \hat{z}_{i+1}^{-1} w^{-\frac{1}{N_f}} \right)^{m_i} \left\{ a_i^{-\frac{1}{4} + \frac{1}{2}(\hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm})} \right. \\ &\quad \times \left. \frac{(\hat{z}_i^{-1} \hat{z}_{i+1} w^{\frac{1}{N_f}} a_i^{\frac{1}{2}} q^{\frac{1}{2} + \hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm}}, \hat{z}_i \hat{z}_{i+1}^{-1} w^{-\frac{1}{N_f}} a_i^{\frac{1}{2}} q^{\frac{1}{2} + \hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm}}; q^2)_{\infty}}{(\hat{z}_i \hat{z}_{i+1}^{-1} w^{-\frac{1}{N_f}} a_i^{-\frac{1}{2}} q^{\frac{1}{2} + \hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm}}, \hat{z}_i^{-1} \hat{z}_{i+1} w^{\frac{1}{N_f}} a_i^{-\frac{1}{2}} q^{\frac{1}{2} + \hat{s}_i^{\pm} - \hat{s}_{i+1}^{\pm} - \frac{1}{N_f} m_J^{\pm}}; q^2)_{\infty}} \right\}. \end{aligned} \quad (4.29)$$

In this expression, one can see the correct contributions of bifundamental and adjoint fields attached to suitable nodes in the quiver diagram.

As a result, the other example of $\mathcal{N} = 2$ multi-flavor mirror symmetry is described as the equality between the indices (4.22) for $\mathcal{CP}\text{-SQED}_{N_f}$ and (4.29) for $\mathcal{P}\text{-U}(1)^{N_f-1}$.

Table 2 briefs the symmetries with variables in these theories. Again, though the global symmetries on one side correspond one-to-one to these on the other side, mirror symmetry accompanies the interchange of the parity type of the gauge symmetries in two theories.

5 Conclusion and outlook

In this paper, we systematically construct Abelian mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$ with the insertions of loop operators by consistently defining the S-operation and general N_f flavors. The unorientable structure of \mathbb{RP}^2 is translated into mod 2 delta functions which are analogies to the Fourier transformations of the usual delta functions on an orientable manifold. The characteristic aspect of our mirror symmetry implemented here is to interchange the two types of the parity conditions under it just as one can see in table 1, 2 for gauge symmetries of the multi-flavor theories and (3.14), (3.15) for the map of the loop operators.

Although we could construct consistent physical dualities on the unorientable manifold, yet there are several interesting questions which we would like to approach in the near future as listed below.

T-operation (Chern-Simons term). Let us comment on defining the T-operation of a $SL(2, \mathbb{Z})$ group [12]. The T-operation T generates the action of adding the Chern-Simons (CS) term of a background gauge field A at the level $k = 1$ to the original Lagrangian $\mathcal{L}(\Phi)$ coupled with it:

$$T : \int \mathcal{D}\Phi e^{\int \mathcal{L}(\Phi) + \int J(\Phi)A} \rightarrow \int \mathcal{D}\Phi e^{\int \mathcal{L}(\Phi) + \int J(\Phi)A + \frac{1}{4\pi} \int A \wedge dA}. \quad (5.1)$$

Therefore, we need to have the CS term compatible under the parity action in order to set the T-operation on $\mathbb{RP}^2 \times \mathbb{S}^1$ as well as the BF term for the S-operation. However, it seems impossible to obtain such T-operation because the CS term made of either a \mathcal{P} -type or a \mathcal{CP} -type vector multiplet becomes parity-odd. At the present moment, there is no appropriate complement to make it parity-even.

Parity anomaly. It might be related to the above comment on the T-operation, if we consider one matter multiplet coupled to a vector multiplet, then, the half-integer level Chern-Simons term for the vector multiplet emerges usually. We do have such situation for the \mathcal{P} -type vector multiplet:

$$\boxed{\Phi} \cdots \rightarrow s^{\pm}, \theta = \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{i\theta}{4}} \right)^{+1} \frac{(e^{-i\theta} q^{\frac{(2-\Delta)}{2}}; q^2)_{\infty}}{(e^{+i\theta} q^{\frac{(0+\Delta)}{2}}; q^2)_{\infty}} & \text{for } (-1)^{s^{\pm}} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{i\theta}{4}} \right)^{-1} \frac{(e^{-i\theta} q^{\frac{(4-\Delta)}{2}}; q^2)_{\infty}}{(e^{+i\theta} q^{\frac{(2+\Delta)}{2}}; q^2)_{\infty}} & \text{for } (-1)^{s^{\pm}} = -1, \end{cases} \quad (5.2)$$

which is obtained by taking $\mathbf{Q}_1 = 0$ and $\mathbf{Q}_2 = 1$ in (2.13). The factors $e^{\pm \frac{i\theta}{4}}$ appearing in the so-called Casimir energy sector are indeed very similar to the Chern-Simons term contribution in the usual index [10]. We have no physical explanation for this “anomalous factor” which is cancelled in all our examples with the gauge symmetry. For a global symmetry, however, our check is verified under the existence of this factor as a fugacity,

hence it should have a certain physical meaning, but we now do not have any plausible interpretation. It may be interesting to study this factor.

3d-3d correspondence. The geometrical aspects of the $SL(2, \mathbb{Z})$ action are naturally encoded in the 3d-3d correspondence [14, 15] which is one interesting application of the index on $\mathbb{RP}^2 \times \mathbb{S}^1$. The fundamental observation to establish this correspondence is a duality

$$ST \circ \mathcal{T}_1 \simeq \mathcal{T}_1, \quad (5.3)$$

where \mathcal{T}_1 is the theory containing a free chiral multiplet (of charge +1) and the CS term of a background field at the level $k = -\frac{1}{2}$. This is derived from $\mathcal{N} = 2$ mirror symmetry with $N_f = 1$, and of course, we can directly check this statement by utilizing the superconformal indices. Clearly, to discuss the 3d-3d correspondence for the theories on $\mathbb{RP}^2 \times \mathbb{S}^1$ based on this viewpoint, the CS term is needed to define in the parity-even way under the parity condition on our geometry. Also, we must find the correct action of the T-operation on the index consistent with parity on \mathbb{RP}^2 . Now, we do not have any resolution and would like to keep approaching this problem.

Factorization. Another recent development in supersymmetric gauge theories on 3d curved manifolds is so-called *factorization* [16–19], which is to factorize exact partition functions on such manifolds into basic building blocks called *holomorphic blocks*. In fact, this property is related to some basic arguments to realize the 3d-3d correspondence. Thus, to find the factorised form of our indices on $\mathbb{RP}^2 \times \mathbb{S}^1$ might give us somewhat important points to establish the 3d-3d correspondence.

More general quiver gauge theories. It is known that 3d Abelian mirror symmetry can be realized for more general quiver theories containing a number of matters charged under gauge symmetries. For instance, an $U(1)^k$ gauge theory with N matter multiplets Φ_i ($i = 1, \dots, N$) is mirror to an $U(1)^{N-k}$ gauge theory with N matter multiplets $\hat{\Phi}_j$ ($j = 1, \dots, N$) [4, 20]. In the former, the matters Φ_i have charges Q_i^a ($a = 1, \dots, k$) under the a -th $U(1)$ gauge symmetry, and the Chern-Simons couplings are expressed by $k^{ab} = \frac{1}{2} \sum_{i=1}^N Q_i^a Q_i^b$. In the latter, the matters $\hat{\Phi}_j$ have charges \hat{Q}_j^p ($p = 1, \dots, N - k$) under the p -th $U(1)$ gauge symmetry, and the Chern-Simons couplings are given by $\hat{k}^{pq} = -\frac{1}{2} \sum_{j=1}^N \hat{Q}_j^p \hat{Q}_j^q$. We are wondering whether we can construct such kinds of quiver theories showing mirror symmetry by appropriately connecting the \mathcal{P} - and \mathcal{CP} -type gauge symmetries.

Non-Abelian gauge theories. In addition, an extension which we should consider is to derive the index formulas of non-Abelian gauge symmetries. The difficulty to do this is lying on determining the Jacobian which arises from projecting the integration measure of the vector multiplet onto the Cartan subalgebra. The solution to this question requires us to find the precise form, especially, of the holonomy on \mathbb{RP}^2 . We hope that these will become clear and provide some new insights into the theories on unorientable manifolds.

Brane construction. Besides the above viewpoints on the quantum field theories, it is a natural question if the brane construction in type IIB string theory [21] can be constructed for mirror symmetry even on $\mathbb{RP}^2 \times \mathbb{S}^1$. Two theories mirror to each other realized as the worldvolume theories on D3-branes are connected by S-duality. Since the key observation of mirror symmetry on $\mathbb{RP}^2 \times \mathbb{S}^1$ is the interchange of \mathcal{P} -type on one side with \mathcal{CP} -type on the other side, we expect that this phenomena in the field theory could be accomplished by S-duality in string theory.

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A BPS loop operators on $\mathbb{RP}^2 \times \mathbb{S}^1$

Before proceeding to the discussion of loop operators, we note the geometrical data. The vielbein for the round two-sphere is taken to be

$$e^1 = d\vartheta, \quad e^2 = \sin \vartheta d\varphi, \quad e^3 = dy. \quad (\text{A.1})$$

The gamma matrices are denoted by $\gamma^a = \sigma^a$ ($a = 1, 2, 3$) where σ^a s are Pauli matrices, and the Killing spinors are given by

$$\epsilon = e^{\frac{1}{2}(y+i\varphi)} \begin{pmatrix} \cos \frac{\vartheta}{2} \\ \sin \frac{\vartheta}{2} \end{pmatrix}, \quad \bar{\epsilon} = e^{\frac{-1}{2}(y+i\varphi)} \begin{pmatrix} \sin \frac{\vartheta}{2} \\ \cos \frac{\vartheta}{2} \end{pmatrix}, \quad (\text{A.2})$$

and we can easily derive their bilinear forms as

$$(\bar{\epsilon}\gamma_\mu\epsilon) = (0, i\sin^2 \vartheta, -1), \quad (\bar{\epsilon}\gamma_\mu\bar{\epsilon}) = e^{-(y+i\varphi)} (-\cos \vartheta, i\sin \vartheta, -\sin \vartheta), \quad (\bar{\epsilon}\epsilon) = \cos \vartheta, \quad (\text{A.3})$$

where $v_\mu = (v_\vartheta, v_\varphi, v_y)$.

A.1 Wilson loop

BPS condition. Let us consider the Wilson loop along a curve C charged under a U(1) gauge symmetry,

$$W_e^{(\pm)}(C) := e^{ie \int_C (A_\mu \dot{x}^\mu \pm i|\dot{x}|\sigma) d\tau}, \quad (\text{A.4})$$

where e is an electric charge. The BPS condition, $\delta_{\bar{\epsilon}} W_e^{(\pm)}(C) = 0$, which is necessary to apply localization technique (see for example [22]), turns to be

$$(\bar{\epsilon}\gamma_\mu\epsilon)\dot{x}^\mu \pm (\bar{\epsilon}\epsilon)|\dot{x}| = 0, \quad (\bar{\epsilon}\gamma_\mu\bar{\epsilon})\dot{x}^\mu = 0. \quad (\text{A.5})$$

It is found by simple calculation with the Killing spinors (A.2) and the relations (A.3) that the BPS Wilson loops wrap the \mathbb{S}^1 direction with sitting on the poles ($\vartheta = 0, \pi$) of the two-sphere. We express these configurations in terms of the curve C as

$$C|_{\vartheta=0} : x^\mu = (0, 0, \pm y), \quad (\text{A.6})$$

$$C|_{\vartheta=\pi} : x^\mu = (0, 0, \mp y), \quad (\text{A.7})$$

and we get the following BPS Wilson loop localized on each pole:

$$W_e^{(\pm)}(C|_{\vartheta=0}) = e^{\pm ie \int_0^{2\pi} dy (A_y + i\sigma)|_{\vartheta=0}}, \quad (\text{A.8})$$

$$W_e^{(\pm)}(C|_{\vartheta=\pi}) = e^{\mp ie \int_0^{2\pi} dy (A_y - i\sigma)|_{\vartheta=\pi}}. \quad (\text{A.9})$$

To argue concretely the loop operators respecting the \mathbb{Z}_2 -invariance on \mathbb{RP}^2 , we here recall the relevant parts of the \mathcal{P} - and \mathcal{CP} -type parity condition for the vector multiplet (see [6] for the complete list):

$\mathcal{P}\text{-type vector multiplet } V^{(\mathcal{P})}$	
$A_y^{(\mathcal{P})} \rightarrow +A_y^{(\mathcal{P})}, \quad \sigma^{(\mathcal{P})} \rightarrow -\sigma^{(\mathcal{P})}.$	(A.10)
Locus: $A_y^{(\mathcal{P})} = \frac{\theta}{2\pi}, \quad \sigma^{(\mathcal{P})} = 0.$	

$\mathcal{CP}\text{-type vector multiplet } V^{(\mathcal{CP})}$	
$A_y^{(\mathcal{CP})} \rightarrow -A_y^{(\mathcal{CP})}, \quad \sigma^{(\mathcal{CP})} \rightarrow +\sigma^{(\mathcal{CP})}.$	(A.11)
Locus: $A_y^{(\mathcal{CP})} = \frac{\theta_\pm}{2\pi}, \quad \sigma^{(\mathcal{CP})} = -s.$	

\mathcal{P} -type Wilson loops. Under the \mathcal{P} -type parity condition, the linear combination of A_y and σ is transformed as

$$(A_y^{(\mathcal{P})} + i\sigma^{(\mathcal{P})}) \rightarrow (A_y^{(\mathcal{P})} - i\sigma^{(\mathcal{P})}). \quad (\text{A.12})$$

Thus, in order to make the operator insertion invariant under parity, it is plausible that the Wilson loop defined by $V^{(\mathcal{P})}$ is taken to be a hybrid of $W_e^{(+)}(C|_{\vartheta=0})$ and $W_e^{(-)}(C|_{\vartheta=\pi})$,

$$\begin{aligned} W_e^{(\mathcal{P})} &= W_e^{(+)}(C|_{\vartheta=0}) \cdot W_e^{(-)}(C|_{\vartheta=\pi}) \\ &= \exp \left[2ie \int_0^{2\pi} dy A_y^{(\mathcal{P})}|_{\vartheta=0, \pi} \right] \\ &\xrightarrow{\text{Locus}} e^{2ie\theta}. \end{aligned} \quad (\text{A.13})$$

We can interpret it from the index point of view (2.17) just by ungauging it, and we denote this as somewhat “trivial” insertion of the Wilson loop associated with a \mathcal{P} -type global symmetry as if it is an operation on the index with background:

$$\hat{w}_e \cdot \mathcal{I}(q; s^\pm, \theta) := e^{2ie\theta} \mathcal{I}(q; s^\pm, \theta). \quad (\text{A.14})$$

Then, the gauge Wilson loop insertion associated with the \mathcal{P} -type dynamical vector multiplet is represented by

$$\hat{W}_e \cdot \mathcal{I}(q) = q^{+\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{s^\pm=0,1} \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \hat{w}_e \cdot \mathcal{I}(q; s^\pm, \theta). \quad (\text{A.15})$$

\mathcal{CP} -type Wilson loops. In contrast, the linear combination of A_y and σ is transformed under the \mathcal{CP} -type parity condition such that

$$(A_y^{(\mathcal{CP})} + i\sigma^{(\mathcal{CP})}) \rightarrow -(A_y^{(\mathcal{CP})} - i\sigma^{(\mathcal{CP})}). \quad (\text{A.16})$$

This observation implies that the Wilson loop defined by $V^{(\mathcal{CP})}$ should be comprised⁵ of $W_{ie/2}^{(+)}(C|_{\vartheta=0})$ and $W_{-ie/2}^{(-)}(C|_{\vartheta=\pi})$ to cancel the nontrivial change of the sign,

$$\begin{aligned} W_e^{(\mathcal{CP})} &= W_{ie/2}^{(+)}(C|_{\vartheta=0}) \cdot W_{-ie/2}^{(-)}(C|_{\vartheta=\pi}) \\ &= \exp \left[-ie \int_0^{2\pi} dy \, \sigma^{(\mathcal{CP})}|_{\vartheta=0,\pi} \right] \\ &\xrightarrow{\text{Locus}} e^{2\pi i e s}. \end{aligned} \quad (\text{A.17})$$

We can interpret it from the index point of view (2.18) just by ungauging it, and we again denote this as somewhat “trivial” insertion of the Wilson loop associated with a \mathcal{CP} -type global symmetry as if it is an operation on the index with background:

$$\hat{w}_e \cdot \mathcal{I}(q; s, \theta_{\pm}) = e^{2\pi i e s} \mathcal{I}(q; s, \theta_{\pm}). \quad (\text{A.18})$$

Then, the gauge Wilson loop insertion associated with \mathcal{CP} -type dynamical vector multiplet is represented by

$$\hat{W}_e \cdot \mathcal{I}(q) = q^{-\frac{1}{8}} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{s \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_{\pm}=0,\pi} \hat{w}_e \cdot \mathcal{I}(q; s, \theta_{\pm}). \quad (\text{A.19})$$

S-transform point of view. More generically, we can turn on the background field coupled to a topological symmetry when we consider the gauge theory, then the relations (A.15) and (A.19) are promoted to the following relation:

$$\hat{W}_e = \hat{S} \hat{w}_e. \quad (\text{A.20})$$

A.2 Vortex loop

Definition via S-operation. As studied in [8], a vortex loop operator associated with a global symmetry can be defined by using the S-operation which we explain in section 2.2,

$$\hat{v}_m = \hat{S}^{-1} \hat{w}_m \hat{S}, \quad (\text{A.21})$$

where m is a magnetic charge of the gauge symmetry. This means that the BPS vortex loop also extends to \mathbb{S}^1 and is located at the poles of the two-sphere, which has been directly derived from the boundary condition preserving supersymmetry in [8, 9]. Now, it is quite convenient to accept this definition in constructing the \mathbb{Z}_2 -invariant vortex loops on $\mathbb{RP}^2 \times \mathbb{S}^1$ because we already saw that the S-operation does work well.

For the vortex loop associated with the gauge symmetry is defined by

$$\hat{V}_m = \hat{S} \hat{v}_m \quad (\text{A.22})$$

as observed in the Wilson loop (A.20). By this construction, the vortex loop automatically satisfies \mathbb{Z}_2 -invariance and the BPS condition. Thus, we focus on the vortex loop for the global symmetry here.

⁵For later use, we choose a *pure imaginary half-integer* charge.

\mathcal{P} -type vortex loops. The \mathcal{P} -type vortex loop operator acts on the index (2.17) coupled to a \mathcal{P} -type global symmetry,

$$\hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta). \quad (\text{A.23})$$

We can translate this in terms of the \mathcal{CP} -type Wilson loop (A.17) by using (A.21), and it is evaluated with the rules of the S-operation (2.19) so that the insertion of v_m changes the Wilson line phase θ by 2π times the magnetic charge,

$$\begin{aligned} & \hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta) \\ &= \left(\dots \longrightarrow \textcircled{\mathcal{P}} \xrightarrow{\text{BF}} \textcircled{\mathcal{CP}} \xrightarrow[-W_m]{-\text{BF}} s^\pm, \theta \right) \\ &= \underbrace{\sum_{s_1^\pm} \int_0^{2\pi} \frac{d\theta_1}{2\pi} q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \mathcal{I}^{\mathbb{RP}^2}(q; s_1^\pm, \theta_1)}_{\dots \longrightarrow \textcircled{\mathcal{P}}} \underbrace{\sum_{s_2 \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_{2\pm}} q^{-\frac{1}{8}} \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} e^{is_1^\pm \theta_{2\pm}} e^{is_2 \theta_1}}_{\textcircled{\mathcal{CP}}} \underbrace{e^{is_1^\pm \theta_{2\pm}} e^{is_2 \theta_1}}_{\text{BF}} \underbrace{e^{2\pi i m s_2}}_{W_m^{(\mathcal{CP})}} \underbrace{e^{-is^\pm \theta_{2\pm}} e^{-is_2 \theta}}_{-\text{BF}} \\ &= \sum_{s_1^\pm} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \mathcal{I}^{\mathbb{RP}^2}(q; s_1^\pm, \theta_1) \left\{ \sum_{s_2 \in \mathbb{Z}} e^{is_2(\theta_1 - \theta + 2\pi m)} \right\} \left\{ \frac{1}{2} \sum_{\theta_{2\pm}} e^{i(s_1^\pm - s^\pm)\theta_{2\pm}} \right\} \\ &= \sum_{s_1^\pm} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \mathcal{I}^{\mathbb{RP}^2}(q; s_1^\pm, \theta_1) \left\{ 2\pi \delta(\theta_1 - \theta + 2\pi m) \right\} \left\{ \delta_{s_1^\pm, s^\pm}^{\text{mod } 2} \right\} \\ &= \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta - 2\pi m) \\ &= e^{-2\pi m \frac{\partial}{\partial \theta}} \mathcal{I}^{\mathbb{RP}^2}(q; s^\pm, \theta). \end{aligned} \quad (\text{A.24})$$

Note that, if $m \in \mathbb{Z}$, it is identical to the original index because θ is defined modulo 2π . To make it easy to use this insertion in our applications, we write the insertion of the \mathcal{P} -type vortex loop with a $U(1)$ global symmetry as the second quiver diagram in (3.7).

\mathcal{CP} -type vortex loops. Further, we can take the vortex loop acting on the index (2.18),

$$\hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta^\pm). \quad (\text{A.25})$$

Unlike the previous case, the definition (A.21) with (A.13) and repeating (2.19) leads to the exact action that the magnetic flux s is shifted by a doubled magnetic charge:

$$\begin{aligned} & \hat{v}_m \cdot \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta^\pm) \\ &= \left(\dots \Longleftrightarrow \textcircled{\mathcal{CP}} \xrightarrow{\text{BF}} \textcircled{\mathcal{P}} \xrightarrow[-W_m]{-\text{BF}} s, \theta^\pm \right) \\ &= \underbrace{\sum_{s_1 \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_1^\pm} q^{-\frac{1}{8}} \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \mathcal{I}^{\mathbb{RP}^2}(q; s_1, \theta_1^\pm)}_{\dots \Longleftrightarrow \textcircled{\mathcal{CP}}} \underbrace{\sum_{s_2^\pm} \int_0^{2\pi} \frac{d\theta_2}{2\pi} q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} e^{is_2^\pm \theta_1 \pm} e^{is_1 \theta_2}}_{\textcircled{\mathcal{P}}} \underbrace{e^{is_2^\pm \theta_1 \pm} e^{is_1 \theta_2}}_{\text{BF}} \underbrace{e^{2im\theta_2}}_{W_m^{(\mathcal{P})}} \underbrace{e^{-is_2^\pm \theta \pm} e^{-is\theta_2}}_{-\text{BF}} \\ &= \sum_{s_1 \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_1^\pm} \mathcal{I}^{\mathbb{RP}^2}(q; s_1, \theta_{1\pm}) \left\{ \sum_{s_2^\pm} e^{is_2^\pm(\theta_{1\pm} - \theta \pm)} \right\} \left\{ \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{i(s_1 + 2m - s)\theta_2} \right\} \\ &= \sum_{s_1 \in \mathbb{Z}} \frac{1}{2} \sum_{\theta_{1\pm}} \mathcal{I}^{\mathbb{RP}^2}(q; s_1, \theta_{1\pm}) \left\{ 2\delta^{\text{mod } 2}(\theta_{1\pm} - \theta \pm) \right\} \left\{ \delta_{s_1, s - 2m} \right\} \\ &= \mathcal{I}^{\mathbb{RP}^2}(q; s - 2m, \theta_\pm) \\ &= e^{-2m \frac{\partial}{\partial s}} \mathcal{I}^{\mathbb{RP}^2}(q; s, \theta_\pm). \end{aligned} \quad (\text{A.26})$$

In conclusion, the quiver rule for this v_m insertion is shown as the second diagram in (3.8).

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